“Angular” matrix integrals

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A. Prats Ferrer, B. Eynard, P. Di Francesco, J.-B. Z.. “*Correlation Functions of*

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Matrix integrals

$$Z_G = \int_G D\Omega \, \exp N\beta \, \Re e \, (\mathrm{tr}\,(\Omega J)) \tag{1}$$

$$Z^{(G)} = \int_G D\Omega \, \exp N\beta \, \Re e \, (\mathrm{tr}\,(A\Omega B\Omega^\dagger)) \tag{2}$$

over a compact group G, are frequently encountered in physics (and in maths) : "Bessel matrix functions" or "angular matrix integrals".
$G = \mathrm{O}(N), \mathrm{U}(N), \mathrm{Sp}(N)$, with respectively $\beta = 1, 2, 4$.
Invariance under $J \to \Omega_1 J \Omega_2$ and $A \to \Omega_1 A \Omega_1^\dagger$, $B \to \Omega_2 B \Omega_2^\dagger$, resp.
Z_G expressible as a sum of $\prod_i \mathrm{tr}\,(JJ^\dagger)^{p_i}$ and $Z^{(G)}$ as a sum of $\prod_i \mathrm{tr}\, A^{p_i} \prod_j \mathrm{tr}\, B^{q_j}$

Matrix integrals

$$Z_G = \int_G D\Omega \exp N\beta \,\Re e\, (\mathrm{tr}\,(\Omega J)) \qquad (1)$$

$$Z^{(G)} = \int_G D\Omega \exp N\beta \,\Re e\, (\mathrm{tr}\,(A\Omega B\Omega^\dagger)) \qquad (2)$$

over a compact group G, are frequently encountered in physics (and in maths) : “Bessel matrix functions”. Mostly studied for $G = \mathrm{U}(N)$ ($\beta = 2$).

What happens for other groups, e.g. $G = \mathrm{O}(N)$ ($\beta = 1$), $\mathrm{Sp}(N)$ ($\beta = 4$)?

- If A and B are both real skew-symmetric (i.e. in the Lie algebra of $\mathrm{o}(N)$), resp. both quaternionic antiselfdual (in $\mathrm{sp}(N)$), Z is known exactly from the work of Harish-Chandra ’57. Also correlation functions are known [Eynard *et al*].
- If A and B are both real symmetric, resp. both quat. selfdual, much more complicated and elusive, [Brézin & Hikami ’02-06, Bergère & Eynard 08].
- if they are neither, . . .?
- Expect simplification as $N \to \infty$ [Weingarten ’78]. Universality of (1), (2).

1. The Harish-Chandra integral. [Harish-Chandra 1957]

For A and B in the *Lie algebra* $\mathfrak{g}$ of G, in fact in a *Cartan* algebra

$$Z^{(G)} = \int_G D\Omega \exp N\beta \operatorname{tr}(A\Omega B\Omega^\dagger) = \text{const.} \sum_{w \in W} \frac{\exp N\beta \operatorname{tr} AB^w}{\Delta_G(A)\Delta_G(B^w)} \qquad (3)$$

$\Delta_G(A) := \prod_{\alpha>0} \langle \alpha, A \rangle$, a product over the positive roots, W the Weyl group.

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[illegible]

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$\Delta_G(A) := \prod_{\alpha>0} \langle \alpha, A \rangle$, a product over the positive roots, W the Weyl group.

More concretely, for $G = U(N)$, take $A = \operatorname{diag}(a_i)$, $B = \operatorname{diag}(b_i)$

$$Z^{(U)} = \text{const.} \frac{\det e^{\beta N a_i b_j}}{\prod_{i<j}(a_i - a_j)(b_i - b_j)}$$ [Itzykson-Z '80]

and for $G =$

Proofs of this H-C formula

– *Heat kernel*

$Z = t^{-\frac{1}{2}\dim G} \int_G D\Omega\, e^{-\frac{1}{2t}N\beta\, \mathrm{tr}\,(A-\Omega B\Omega^\dagger)^2}$ satisfies $(N\beta\frac{\partial}{\partial t} - \frac{1}{2}\Box_A^2)Z = 0$ and boundary cond $Z \underset{t\to 0}{\longrightarrow} \mathrm{const} \int_G d\Omega\, \delta(A - \Omega B\Omega^\dagger)$. Rewrite in “radial coordinates” a_i using the expression of the Laplacian

$$\Box_A^2 = \Delta_G^{-2}(A) \sum_i \partial_i \Delta_G^2 \partial_i$$

Correlation functions

What about the associated “correlation functions” of invariant traces

$$\int D\Omega \, e^{-\mathrm{tr} A\Omega B\Omega^\dagger} \prod \mathrm{tr}\,(A^{p_1}\Omega B^{q_1}\Omega^\dagger A^{p_2} e^{D} \;\; e$$

Correlation functions

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$$\int D\Omega\, e^{-\mathrm{tr} A\Omega B\Omega^\dagger} \prod \mathrm{tr}\,(A^{p_1}\Omega B^{q_1}\Omega^\dagger A^{p_2}\cdots) \qquad ?$$

(still invariant under $A \to \Omega_1 A \Omega_1^\dagger$, $B \to \Omega_2 B \Omega_2^\dagger$)

Is there still some localization property? Yes !

$$\int D\Omega\, e^{-\mathrm{tr} A\Omega B\Omega^\dagger} F(A, \Omega B\Omega^\dagger) = c_n \sum_{w \in W} \frac{e^{-\mathrm{tr} AB^w}}{\Delta(A)\Delta(B^w)} \int_{\mathfrak{n}_+ = [\mathfrak{b},\mathfrak{b}]} DT\, e^{-\mathrm{tr}\, TT^\dagger} F(A+T, B^w + T^\dagger)$$

2. The integral (2) in the symmetric case

$$Z^{(G)} = \int_G D\Omega \exp N\beta \operatorname{tr}(A\Omega B\Omega^\dagger)$$

for $A = A^\dagger$ and $B = B^\dagger$.

For $G = U(N)$, A and B hermitian rather than *anti*hermitian, no difference, HCIZ formula works.

For $G = O(N)$, A and B real symmetric, ??G105.982 ??G105.1898pd[(real)-250(symme10051)-25902Td[(F)15(or)]TJ/F329.962

Many nice features

– finite (semi-classical) expansion and “τ-expansion” for β an

$\sum_k M_{ik} = \sum_i M_{ik} = Z$ and $\sum_j K_{ij} M_{jk} = (N\beta) M_{ik} b_k$. Can iterate that equation to get

$$\sum_j K^p_{ij} M_{jk} = M_{ik} (N\beta)^p b^p_k$$

and summing over i and k

$$\underbrace{\Big(\sum_{ij} K^p_{ij}\Big)}_{\text{a differential operator of order } p} Z = (N\beta)^p \operatorname{tr} B^p Z \,. \qquad (7)$$

Two remarks

1. *This solves the following problem :*

Define the differential operator $D_p(\partial/\partial A)$ by
$D_p(\partial/\partial A) e^{N \operatorname{tr} AB} = N^p \operatorname{tr} B^p e^{N \operatorname{tr} AB}$
If D_p acts on *invariant functions* $F(A) = F(\Omega A \Omega^\dagger)$, how to write it in terms

of $\partial/\partial a$

3. Large N limit

Expect things to simplify as $N \to \infty$ [Weingarten '78]. Look at the "free energies" :

$$W_G(J.J^\dagger) = \lim_{N\to\infty} \frac{1}{N^2} \log Z_G$$

and

$$F_G(A,B) = \lim_{N\to\infty} \frac{1}{N^2} \log Z^{(G)}$$

Then $W(X)$ and $F(A,B)$ are, *up to an overall factor*, independent of $G = \mathrm{O}(N),\ \mathrm{U}(N)$!

(Not true at finite N !)

More precisely,

$$W_O(J.J^\dagger) = 1$$

For $Z_{\mathrm{O}} = \int_{\mathrm{O}(N)} DO \exp N \mathrm{tr}\,(J.O)$, follow the steps of [Brézin-Gross '80]:
the trivial identity $\sum_j \frac{\partial^2 Z_{\mathrm{O}}}{\partial J_{ij} \partial J_{kj}} = N^2 \delta_{ik} Z_{\mathrm{O}}$ is reexpressed in terms of the eigenvalues λ_i of the real symmetric matrix $J.J^t$:

$$4\lambda_i \frac{\partial^2 Z_{\mathrm{O}}}{\partial \lambda_i^2} + \sum_{j=i} \frac{2\lambda_j}{\lambda_j - \lambda_i} \left(\frac{\partial Z_{\mathrm{O}}}{\partial \lambda_j} - \frac{\partial Z_{\mathrm{O}}}{\partial \lambda_i} \right) +$$

For $Z^{(\mathrm{O})} = \int_{\mathrm{O}(N)} DO \exp N \mathrm{tr}\,(AOBO^t)$, take A and B both skew-symmetric, or both symmetric.

- A and B both skew-symmetric [Harish-Chandra]

block-diagonal form $A = \mathrm{diag}\left(\begin{pmatrix} 0 & a_i \\ -a_i & 0 \end{pmatrix}_{i=1,\cdots,m}\right)$, B likewise, recall

$$Z^{(\mathrm{O})} = \mathrm{const.}\frac{\det(2\cosh 2Na_i b_j)}{\Delta_{\mathrm{O}}(a)\Delta_{\mathrm{O}}(b)}$$

(for $\mathrm{O}(N = 2m)$), with $\Delta_{\mathrm{O}}(a) = \prod_{1\le i<j\le m}(a_i^2 - a_j^2)$.

Regard A as $N \times N$ anti-Hermitian, eigenvalues $A_j = \pm i a_j$, B likewise. Easy to check that as $N \to \infty$,

$$Z^{(\mathrm{U})}(A,B) = \frac{\det\left(e^{2NA_iB_j}\right)}{\Delta(A)\Delta(B)} \approx \left(\frac{(\det(e^{2Na_ib_j})_{1\le i,j\le m}}{\Delta_{\mathrm{O}}(a)\Delta_{\mathrm{O}}(b)}\right)^2 = (Z^{(\mathrm{O})}(A,B))^2$$

- *A* and *B* both symmetric

Can take them in diagonal form $A = \mathrm{diag}\, a_i$, $B = \mathrm{diag}\, b_i$

Then Bergère-Eynard equation $D_p Z = (N\beta)^p \mathrm{tr}\, B^p Z$ (7), in the large N limit, yields

$$\sum_i \left(\frac{N}{\beta} \frac{\partial F^{(G)}}{\partial a_i} + \frac{1}{2N} \sum_{j=i} \frac{1}{a_i - a_j} \right)^p = \mathrm{tr}\, B^p \qquad (11)$$

Hence $F^{(O)}$ ($\beta = 1$) satisfies same set of equations as $\frac{1}{2} F^{(U)}$ ($\beta = 2$), QED.

Particular case where A is of finite *rank* r. Then in the expansion of
$F = \sum_{p,q} \prod(\frac{1}{N}\mathrm{tr}A^{p_i})\prod(\frac{1}{N}\mathrm{tr}B^{q_j})$, terms with a single trace of A dominate.

In the U(N) case (and $N \to \infty$) ([IZ '80])

$$F^{(\mathrm{U})} \quad \sum_{p=1}^{\infty} \frac{1}{p}(\frac{1}{N}\mathrm{tr}A^p)\psi_p(B)$$

where $\psi_p(B) = p$-th "non-crossing cumulant" of B

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Spin glass Hamiltonian with n replicas of N Ising spins

$$H = \sum_{i,j=1}^{N} \underbrace{\sum_{a=1}^{n} \sigma_i^a \sigma_j^a}_{\Omega_{ij}} O_{ij} \qquad \Omega \text{ of rank } \leq n$$

with a coupling O_{ij}, a real, orthogonal, symmetric matrix with an equal number of ± 1 eigenvalues, $O = V^t.D.V$.

Have to compute $Z = \int_{O(N)} dV \exp \beta \operatorname{tr} DV\Omega V^t$.

Now according to Marinari, Parisi, Ritort, pretend you integrate over the unitary group,
compute $\sum \frac{1}{p} \operatorname{tr} \Omega^p \psi_p(D) =: \operatorname{tr} G(\Omega)$

and (with some insight ...) the correct formula is $\frac{1}{2} G(2\Omega)$! ...

Proved later by Collins, Collins and Sniady, Guionnet & Maida

Conclusion and Open issues

– More explicit formulae for Z, F
– A priori argument for universality, graphical argument ?
– Relations with integrability: D-H localization, finite semi-classical expansions, Calogero, . . .